Analytic Function

A **single-valued function** is **function** that, for each point in the domain, has a unique **value** in the range. A **single-valued** complex **function** of a complex variable is a complex **function** that has the same **value** at every point.
For each value of x corresponds to one value of f(x).
Let $f(x) = x^2 + 5x - 3$ Then

x	$f(x) = x^2 + 5x - 3$
1	3
2	11

So f(x) is single-valued function for every value of x.

A function f(x) is Multi-valued function that, for each value of x corresponds to two or more values of f(x).
Let $f(x) = \sqrt{x^2}$ Then

x	$f(x) = \sqrt{x^2}$
1	± 1
2	± 2

So f(x) is multi-valued function for every value of x.

Analytic Function

A **single-valued function** f(z) is said to be **analytic** if it is defined and differentiable at each point in a region **R** (i.e. $\mathbf{f'(z)}$ exists in the region). This function is also called regular function or holomorphic function.

Necessary condition for analytic function

Consider $\mathbf{f(z) = u(x, y) + iv(x, y)}$ be analytic in a region R then necessary condition is that it satisfy the **Cauchy-Reimann** equations $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and $\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$ where $\frac{\delta u}{\delta x}, \frac{\delta v}{\delta y}, \frac{\delta v}{\delta x}$ and $\frac{\delta u}{\delta y}$ are four partial derivatives continuous in the region R.

❖ **Prove that $f(z) = z|z|$ is not analytic.**

Solution: Consider

$$f(z) = u(x,y) + iv(x,y) = z|z|$$

We have

$$z = x + iy \text{ then } |z| = \sqrt{x^2 + y^2}$$

So $f(z) = u(x,y) + iv(x,y)$

$$= (x + iy)\sqrt{x^2 + y^2}$$

$$= x\sqrt{x^2 + y^2} + iy\sqrt{x^2 + y^2}$$

Comparing real and imaginary part we have

$u(x,y) = x\sqrt{x^2 + y^2}$ **and** $v(x,y) = y\sqrt{x^2 + y^2}$

Now

$$\frac{\delta u}{\delta x} = \frac{\delta}{\delta x}\left(x\sqrt{x^2 + y^2}\right)$$

$$= \sqrt{x^2 + y^2}\ \frac{\delta}{\delta x}(x) + x\frac{\delta}{\delta x}\left(\sqrt{x^2 + y^2}\right)$$

$$= \sqrt{x^2 + y^2} + x\frac{1}{2\sqrt{x^2+y^2}}\frac{\delta}{\delta x}(x^2 + y^2)$$

$$= \sqrt{x^2 + y^2} + x\frac{1}{2\sqrt{x^2+y^2}}.2x$$

$$= \sqrt{x^2 + y^2} + \frac{x^2}{\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}}$$

$$= \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

$$\frac{\delta u}{\delta y} = \frac{\delta}{\delta y}\left(x\sqrt{x^2 + y^2}\right)$$

$$= x\frac{\delta}{\delta y}\left(\sqrt{x^2 + y^2}\right)$$

$$= x\frac{1}{2\sqrt{x^2+y^2}}\frac{\delta}{\delta y}(x^2 + y^2)$$

$$= x\frac{1}{2\sqrt{x^2+y^2}}.2y$$

$$= \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\delta v}{\delta x} = \frac{\delta}{\delta x}\left(y\sqrt{x^2 + y^2}\right)$$

$$= y\frac{\delta}{\delta x}\left(\sqrt{x^2 + y^2}\right)$$

$$= y\frac{1}{2\sqrt{x^2+y^2}}\frac{\delta}{\delta x}(x^2 + y^2)$$

$$= x\frac{1}{2\sqrt{x^2+y^2}}.2x$$

$$= \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}\left(y\sqrt{x^2 + y^2}\right)$$

$$= \sqrt{x^2 + y^2}\ \frac{\delta}{\delta y}(y) + y\frac{\delta}{\delta y}\left(\sqrt{x^2 + y^2}\right)$$

$$= \sqrt{x^2 + y^2} + y\frac{1}{2\sqrt{x^2+y^2}}\frac{\delta}{\delta y}(x^2 + y^2)$$

$$= \sqrt{x^2 + y^2} + y\frac{1}{2\sqrt{x^2+y^2}}.2y$$

$$= \sqrt{x^2 + y^2} + \frac{y^2}{\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2+y^2}{\sqrt{x^2+y^2}}$$

$$= \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

Cauchy-Riemann equations $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ **and** $\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$ **are not satisfied.**

So $f(z)$ is not analytic.

❖ Show that $\mathbf{f(z) = e^x(\cos y + i \sin y)}$ **is holomorphic and find its derivative.**

Solution:

Consider

$$f(z) = u(x, y) + iv(x, y)$$
$$= e^x(\cos y + i\sin y)$$

Comparing real and imaginary part we have

$u(x, y) = e^x\cos y$ **and** $v(x, y) = e^x\sin y$

Now

$$\frac{\delta u}{\delta x} = \frac{\delta}{\delta x}(e^x\cos y) = e^x\cos y$$
$$\frac{\delta u}{\delta y} = \frac{\delta}{\delta y}(e^x\cos y) = -e^x\sin y$$
$$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}(e^x\sin y) = e^x\cos y$$
$$\frac{\delta v}{\delta x} = \frac{\delta}{\delta x}(e^x\sin y) = e^x\sin y$$

Cauchy-Riemann equations $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ **and** $\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$ **are satisfied.**

So $f(z)$ **is holomorphic.**

Again

$$f(z) = u(x, y) + iv(x, y)$$

Then

$$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$$
$$= e^x\cos y + ie^x\sin y$$
$$= e^x(\cos y + i\sin y)$$
$$= e^x . e^{iy}$$
$$= e^{x+iy}$$
$$= e^z$$

Example 3 : Use C-R equation concept to find derivative of $f(z) = z^2$.

Solution:- We have $f(z) = z^2 = x^2 - y^2 + i2xy$

$\therefore u(x, y) = x^2 - y^2, v(x, y) = 2xy$

$\therefore u_x = 2x = v_y$ and $u_y = -2y = -v_x$

So, C-R equations are satisfied everywhere in z-plane.

$\therefore f'(z)$ exists everywhere. Thus we get

$$\boxed{f'(z) = u_x + iv_x = v_y - iu_y = 2x + i2y = 2z}$$

Example 4 : Show that neither $f(z) = \bar{z}$ nor $f(z) = |z|$ is an analytic function.

Solution:- We have $f(z) = \bar{z} = x - iy$

$\therefore u(x, y) = x, v(x, y) = -y$

$\therefore u_x = 1, u_y = 0, v_x = 0, v_y = -1$ $\quad \therefore u_x \neq v_y$

So, C-R equation is not satisfied.

$\therefore f(z) = \bar{z}$ is not analytic.

Show that $f(z) = (2x^2 + y) + i(y^2 - x)$ is not analytic at any point.